

FIRST PASSAGE TIMES FOR BIVARIATE DIFFUSION PROCESSES

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We determine the joint distribution of the exit times from a two-dimensional strip of a bivariate diffusion process. We consider two different situations; crossing or absorbing boundaries. In both cases, this distribution depends on the transition density of the process constrained to evolve under the boundary. Solving a two-dimensional Kolmogorov forward equation, we explicitly derive these two quantities for a bivariate Wiener process with drift and non-diagonal covariance matrix. Explicit expressions for other diffusion processes in presence of either absorbing or crossing boundaries are not available. We propose a numerical algorithm, which is shown to be convergent. A comparison between theoretical and numerical results for Wiener and an illustration of the numerical approximation for a bivariate Ornstein-Uhlenbeck process in presence of absorbing boundaries are carried out.

1. Introduction and motivation. The first passage time (FPT) problem for one-dimensional stochastic processes has been widely investigated through simulation, analytical and numerical methods [3, 7, 14, 17]. Besides its mathematical interest, the derivation of the FPT distribution is relevant in different fields, e.g. neuroscience, reliability theory, finance, and epidemiology. In neuroscience, FPTs describe the times when the neuron releases an electrical impulse, called spike. In reliability theory, FPTs model the epochs when a crash of an object happens. In finance, FPTs describe the time when a bond or a stock reaches a certain value and it is profitable to sell or buy. In epidemiology, FPTs describe the times when an epidemic reaches a threshold level, causing a major disease outbreak. Connections between neurons, common shocks and direct interaction between objects, dependencies between stocks in the same portfolio or belonging to the same market and interactions between populations suggest the presence of dependencies between FPTs. Therefore, it is of interest to extend the FPT problem to more general scenarios.

Our aim is to solve the two-dimensional FPT problem of a bivariate diffusion process in presence of crossing or absorbing boundaries, motivated by problems arising in the framework of neuronal network modeling. This paper is inspired by [13], where the FPT of a two-dimensional

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Brownian motion without drift, in presence of absorbing boundaries, was derived. The bivariate Wiener was proposed as an oversimplified model of a neural network. A more realistic model is proposed in [22], where a multivariate Ornstein-Uhlenbeck (OU) process is introduced to describe the dynamics of k neurons. This model is obtained as a diffusion approximation of a multivariate Stein process. The presence of common inputs between neurons determines dependence in their membrane potential evolutions and in their spike times. A bivariate version of this model is investigated through simulations in [21].

For one-dimensional processes, transition densities are known for Wiener, OU and square root processes (also called Cox-Ingersoll-Ross or Feller process), and all those cases where an explicit solution of the Kolmogorov forward equation is available [9, 18, 19]. The transition density of a process in presence of absorbing boundaries and the FPT density are generally unknown, except for Wiener [20] and a special case of the OU process [11]. As an alternative approach, numerical methods can be applied [20].

For multivariate processes, the transition density is known for Gaussian diffusion processes [4]. Furthermore, in [5], the FPT of one component of a bivariate Gaussian diffusion process through a constant boundary is studied. However, neither the transition density in presence of absorbing boundaries, nor the joint FPT density are available. Our aim is to compute them in the two cases of crossing or absorption at the boundaries. In presence of absorbing boundaries, we assume that the first component reaching its threshold, which we call the faster component, is absorbed there, while the other component, which we call the slower component, independently pursues its evolution till the epoch when it attains its boundary. Thus the considered process is a bivariate diffusion until the crossing of the faster component and then it becomes a univariate diffusion. For crossing boundaries, we assume that the faster component pursues its evolution after having reached its threshold, together with the slower component, which evolves until it crosses its boundary.

In Section 2 we introduce notation and mathematical background used throughout the paper. In Section 3, we calculate the joint distribution of the FPTs of a bivariate diffusion process in presence of crossing or absorbing boundaries. In both cases, the distribution depends on the unknown joint probability densities of the slower component constrained to be below its threshold, and of the FPT of the faster component. We show that these unknown conditional densities solve a system of Volterra-Fredholm integral equations. In Section 4 we present a numerical algorithm to solve the system. The obtained numerical solution is then used to evaluate the joint distribution of the FPTs. In Section 5 we study the order of convergence of the error of the proposed algorithm. In Section 6 we determine explicit expressions of the joint distribution of the FPTs of a bivariate Wiener process with constant drifts and non-diagonal covariance matrix. In particular, the transition density of the bivariate Wiener process in presence of absorbing boundary conditions is explicitly derived as solution of a bivariate Kolmogorov forward equation. The results in Section 6 extend and correct those in [6, 12, 13]. Finally, in Section 7 we apply the numerical algorithm to evaluate the joint FPT density of bivariate Wiener and OU processes in presence of absorbing boundaries, comparing numerical and theoretical results for Wiener.

Throughout the paper, bivariate diffusion processes are considered. To apply the algorithm and numerically evaluate the joint FPT density, we are restricted to processes with a known bivariate transition density, e.g. Gaussian diffusion.

2. Mathematical background. Consider a two-dimensional time homogeneous diffusion process $\mathbf{X} = \{(X_1, X_2)(t); t > t_0\}$, originated at time t_0 in $\mathbf{X}(t_0) = \mathbf{x}_0 = (x_{01}, x_{02})$, solution of the stochastic differential equation

$$(2.1) \quad d\mathbf{X}(t) = \boldsymbol{\mu}(\mathbf{X}(t))dt + \boldsymbol{\Sigma}(\mathbf{X}(t))d\mathbf{W}(t).$$

Let $\mathbf{B} = (B_1, B_2) \in \mathbb{R}^2$ be a two-dimensional boundary, with $B_i > x_{0i}, i = 1, 2$ and denote T_i the random variable FPT of X_i through the boundary B_i , defined by

$$T_i = \inf\{t > t_0 : X_i(t) > B_i\} \quad i = 1, 2.$$

Before the first exit time from the strip $(-\infty, B_1) \times (-\infty, B_2)$, the process \mathbf{X} evolves under the boundary \mathbf{B} , and we denote it as \mathbf{X}^a , i.e.

$$\mathbf{X}^a = \{\mathbf{X}(t); t \in [t_0, \min(T_1, T_2)]\}.$$

Similarly, we denote X_i^a the unidimensional process X_i evolving under the boundary B_i , before time T_i , i.e.

$$X_i^a = \{X_i(t); t \in [t_0, T_i]\}, \quad i = 1, 2.$$

If \mathbf{Z} is a k -dimensional process, we denote its transition probability by $F_{\mathbf{Z}}(\mathbf{x}, t | \mathbf{y}, s) = \mathbb{P}(\mathbf{Z}(t) < \mathbf{x} | \mathbf{Z}(s) = \mathbf{y})$, its survival function by $\bar{F}_{\mathbf{Z}}(\mathbf{x}, t | \mathbf{y}, s)$ and the transition probability density function (pdf) by $f_{\mathbf{Z}}(\mathbf{x}, t | \mathbf{y}, s)$ for $s < t, \mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. Throughout the paper, $\mathbf{Z} = \mathbf{X}$ or $\mathbf{Z} = \mathbf{X}^a$, for $k = 1$ or 2 .

For $s < t$, we denote $f_{X_i^a | X_j^a}(x_i, t | x_j, t; \mathbf{y}, s)$ the conditional pdf of X_i^a given X_j^a , for $i, j = 1, 2, i \neq j$, defined by

$$f_{X_i^a | X_j^a}(x_i, t | x_j, t; \mathbf{y}, s) dx_i = \mathbb{P}(X_i^a(t) \in dx_i | X_j^a(t) = x_j, \mathbf{X}^a(s) = \mathbf{y}),$$

and $f_{X_i^a | T_j}(x_i | t; y_i, s)$ the transition pdf of X_i^a conditioned on T_j when X_i^a starts in y_i at time $s < T_j$, defined by

$$(2.2) \quad f_{X_i^a | T_j}(x_i | t; y_i, s) dx_i = \mathbb{P}(X_i^a(T_j) \in dx_i | T_j = t, X_i^a(s) = y_i).$$

The joint pdf of T_j and $X_i^a(T_j)$ when \mathbf{X}^a starts in \mathbf{y} at time $s < T_j$ is denoted by $f_{(X_i^a, T_j)}(x_i, t | \mathbf{y}, s)$ and defined by

$$(2.3) \quad f_{(X_i^a, T_j)}(x_i, t | \mathbf{y}, s) dx_i dt = \mathbb{P}(X_i^a(T_j) \in dx_i, T_j \in dt | \mathbf{X}^a(s) = \mathbf{y}).$$

We denote $f_{(X_i, T_j)}((x_i, B_j), t | (y_i, y_j), s)$ the joint pdf of T_j and $X_i(T_j)$, when X_j starts in y_j , X_i starts in y_i at time s , which is defined by

$$f_{(X_i, T_j)}((x_i, B_j), t | \mathbf{y}, s) dx_i dt = \mathbb{P}(X_i(T_j) \in dx_i, T_j \in dt | \mathbf{X}(s) = \mathbf{y}).$$

The pdf of T_i is defined by $g_{T_i}(t | y_i, s) = \mathbb{P}(T_i \in dt | X_i(s) = y_i)$. Our aim is to determine the joint distribution function of $\mathbf{T} = (T_1, T_2)$ for a process \mathbf{X} starting in $\mathbf{y} < \mathbf{B}$ at time s in presence of either crossing or absorbing boundary \mathbf{B} .

To simplify the notation, throughout we omit the starting position when $\mathbf{y} = \mathbf{x}_0$, and the starting time s when $s = t_0$.

3. Joint distribution of (T_1, T_2) . In presence of absorbing boundaries, the joint distribution of (T_1, T_2) can be expressed in terms of the marginal FPTs densities and of the joint pdfs (2.3) as follows

THEOREM 3.1. *Let \mathbf{X} be a two-dimensional diffusion process with $\mathbf{X}(t_0) = \mathbf{x}_0$ and \mathbf{B} be a two-dimensional absorbing boundary with $B_1 > x_{01}$ and $B_2 > x_{02}$. The joint distribution of (T_1, T_2) is*

$$(3.1) \quad \begin{aligned} & \mathbb{P}(T_1 < t_1, T_2 < t_2) \\ &= \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \int_{-\infty}^{B_j} \left(\int_{s_i}^{t_j} g_{T_j}(s_j | x_j, s_i) ds_j \right) f_{(X_j^a, T_i)}(x_j, s_i) dx_j ds_i. \end{aligned}$$

PROOF.

$$\begin{aligned} & \mathbb{P}(T_1 < t_1, T_2 < t_2) \\ &= \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \mathbb{P}(T_1 < t_1, T_2 < t_2 | T_i < T_j, T_i = s_i) \mathbb{P}(T_i \in ds_i, T_i < T_j). \end{aligned}$$

Conditioning on the value of the component which has not yet reached its boundary, at the time when the other component crosses its boundary, we get

$$\begin{aligned} & \mathbb{P}(T_1 < t_1, T_2 < t_2) \\ &= \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \int_{-\infty}^{B_j} \mathbb{P}(T_j < t_j | T_i = s_i, X_j^a(s_i) = x_j) \mathbb{P}(X_j^a(s_i) \in dx_j | T_i = s_i) \mathbb{P}(T_i \in ds_i, T_i < T_j) \\ &= \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \int_{-\infty}^{B_j} \mathbb{P}(T_j < t_j | X_j^a(s_i) = x_j) \mathbb{P}(X_j^a(s_i) \in dx_j, T_i \in ds_i), \end{aligned}$$

where the last equality holds because \mathbf{X} and thus \mathbf{X}^a are Markov processes. \square

REMARK 3.1. *The expression*

$$(3.2) \quad f_{(X_j^a, T_i)}(x_j, s_i) dx_j ds_i = f_{X_j^a|T_i}(x_j|s_i) g_{T_i}(s_i) dx_j ds_i$$

can be plugged into (3.1), obtaining

$$(3.3) \quad \begin{aligned} & \mathbb{P}(T_1 < t_1, T_2 < t_2) \\ &= \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \int_{-\infty}^{B_j} \left(\int_{s_i}^{t_j} g_{T_j}(s_j|x_j, s_i) ds_j \right) f_{X_j^a|T_i}(x_j|s_i) g_{T_i}(s_i) dx_j ds_i. \end{aligned}$$

This expression is useful when g_{T_i} is known, because it allows to rewrite (3.1) in terms of the unknown function $f_{X_j^a|T_i}$.

From Theorem 3.1, it follows

COROLLARY 3.1. *The joint density of (T_1, T_2) for a two-dimensional diffusion process in presence of an absorbing boundary \mathbf{B} is given by*

$$(3.4) \quad \mathbb{P}(T_1 \in dt_1, T_2 \in dt_2) = \sum_{i,j=1; i \neq j}^2 \int_{-\infty}^{B_j} g_{T_j}(t_j|x_j, t_i) f_{(X_j^a, T_i)}(x_j, t_i) dx_j dt_i dt_j$$

In presence of crossing boundaries, the joint FPT distribution of a diffusion process is given by the following

THEOREM 3.2. *Let \mathbf{X} be a two-dimensional diffusion process with $\mathbf{X}(t_0) = \mathbf{x}_0$ and \mathbf{B} be a two-dimensional crossing boundary with $B_1 > x_{01}$ and $B_2 > x_{02}$. The joint distribution of \mathbf{T} is*

$$(3.5) \quad \begin{aligned} & \mathbb{P}(T_1 < t_1, T_2 < t_2) \\ &= \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \int_{-\infty}^{B_j} \int_{s_i}^{t_j} \int_{-\infty}^{\infty} f_{(X_i, T_j)}((x_i, B_j), s_j | (B_i, x_j), s_i) f_{(X_j^a, T_i)}(x_j, s_i) dx_i ds_j dx_j ds_i. \end{aligned}$$

PROOF. Before the first crossing time from the strip, the bivariate diffusion process behaves as the bivariate diffusion process. Therefore, from Theorem (3.1), it holds

$$(3.6) \quad \mathbb{P}(T_1 < t_1, T_2 < t_2) = \sum_{i,j=1; i \neq j}^2 \int_{t_0}^{t_i} \int_{-\infty}^{B_j} \mathbb{P}(T_j < t_j | T_i = s_i, X_j^a(s_i) = x_j) \mathbb{P}(X_j^a(s_i) \in dx_j, T_i \in ds_i).$$

After the first exit time from the strip, the faster component i pursues its evolution, starting in B_i at time s_i , while the component j starts in $X_j(s_i)$ and has its FPT at time T_j . Therefore, we

have

$$\begin{aligned}
\mathbb{P}(T_j < t_j | T_i = s_i, X_j^a(s_i) = x_j) &= \int_{s_i}^{t_j} \mathbb{P}(T_j \in ds_j | T_i = s_i, X_j^a(s_i) = x_j) \\
&= \int_{s_i}^{t_j} \int_{-\infty}^{\infty} \mathbb{P}(T_j \in ds_j, X_i(s_j) \in x_i | T_i = s_i, X_j^a(s_i) = x_j) \\
(3.7) \quad &= \int_{s_i}^{t_j} \int_{-\infty}^{\infty} \mathbb{P}(T_j \in ds_j, X_i(s_j) \in x_i | X_i(s_i) = B_i, X_j^a(s_i) = x_j)
\end{aligned}$$

where the last equality holds because \mathbf{X} is a Markov process. The theorem follows from plugging (3.7) into (3.6). \square

The density $f_{(X_i, T_j)}$ in (3.5) is not explicitly known, but it can be numerically evaluated as done in [5], provided that the bivariate transition density is known, e.g. for bivariate Gaussian diffusions. Note that the joint FPT distributions (3.1) and (3.5) of a bivariate diffusion process in presence of absorbing and crossing boundaries, respectively, depend on $f_{(X_j^a, T_i)}$. In Section 6, we explicitly derive $f_{(X_j^a, T_i)}$ for a bivariate Wiener. For other processes, the unknown density $f_{(X_j^a, T_i)}$ solves a system of Volterra-Fredholm integral equations [1]:

THEOREM 3.3. *Let \mathbf{X} be a bivariate diffusion process with $\mathbf{X}(t_0) = \mathbf{x}_0$ and let \mathbf{B} be a two-dimensional boundary with $B_1 > x_{01}$ and $B_2 > x_{02}$. The joint transition pdfs $f_{(X_i^a, T_j)}$, for $i, j = 1, 2$; $i \neq j$, are solutions of the following system of Volterra-Fredholm first kind integral equations*

$$\begin{aligned}
\bar{F}_{\mathbf{X}}((x_1, B_2), t) &= \int_{t_0}^t \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((x_1, B_2), t | (B_1, y), \tau) f_{(X_2^a, T_1)}(y, \tau) dy d\tau \\
(3.8a) \quad &+ \int_{t_0}^t \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((x_1, B_2), t | (y, B_2), \tau) f_{(X_1^a, T_2)}(y, \tau) dy d\tau;
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{\mathbf{X}}((B_1, x_2), t) &= \int_{t_0}^t \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((B_1, x_2), t | (B_1, y), \tau) f_{(X_2^a, T_1)}(y, \tau) dy d\tau \\
(3.8b) \quad &+ \int_{t_0}^t \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((B_1, x_2), t | (y, B_2), \tau) f_{(X_1^a, T_2)}(y, \tau) dy d\tau,
\end{aligned}$$

where $x_1 > B_1$ and $x_2 > B_2$.

PROOF. Let us consider the exit times of the process \mathbf{X} . The survival distribution of \mathbf{X} , for

$x_1 > B_1$ and $x_2 > B_2$, is given by

$$\begin{aligned}
\bar{F}_{\mathbf{X}}(\mathbf{x}, t) &= \mathbb{P}(\mathbf{X}(t) \geq \mathbf{x}, T_1 < T_2) + \mathbb{P}(\mathbf{X}(t) \geq \mathbf{x}, T_1 > T_2) \\
&= \int_{t_0}^t \int_{-\infty}^{B_2} \mathbb{P}(\mathbf{X}(t) \geq \mathbf{x}, T_1 < T_2 | T_1 = \tau, X_2(T_1) = y) \\
&\quad \cdot \mathbb{P}(X_2(T_1) \in dy, T_1 \in d\tau, T_1 < T_2) \\
&+ \int_{t_0}^t \int_{-\infty}^{B_1} \mathbb{P}(\mathbf{X}(t) \geq \mathbf{x}, T_1 > T_2 | T_2 = \tau, X_1(T_2) = y) \\
&\quad \cdot \mathbb{P}(X_1(T_2) \in dy, T_2 \in d\tau, T_1 > T_2) \\
&= \int_{t_0}^t \int_{-\infty}^{B_2} \mathbb{P}(\mathbf{X}(t) \geq \mathbf{x} | \mathbf{X}(\tau) = (B_1, y)) f_{(X_2^a, T_1)}(y, \tau) dy d\tau \\
(3.9) \quad &+ \int_{t_0}^t \int_{-\infty}^{B_1} \mathbb{P}(\mathbf{X}(t) \geq \mathbf{x} | \mathbf{X}(\tau) = (y, B_2)) f_{(X_1^a, T_2)}(y, \tau) dy d\tau,
\end{aligned}$$

where the last equality follows from the strong Markov property. Then, the theorem (3.8) follows by choosing $\mathbf{x} = (x_1, B_2)$ and $\mathbf{x} = (B_1, x_2)$, respectively. \square

COROLLARY 3.2. *Differentiating (3.9) with respect to \mathbf{x} and since $\int_{t_0}^{t_i} \int_{-\infty}^{B_2} f_{(X_1^a, T_2)}(y, \tau) dy d\tau = 1$, it follows*

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}, t) &= \int_{t_0}^t \int_{-\infty}^{B_2} f_{\mathbf{X}}(\mathbf{x}, t | (B_1, y), \tau) f_{(X_2^a, T_1)}(y, \tau) dy d\tau \\
&+ \int_{t_0}^t \int_{-\infty}^{B_1} f_{\mathbf{X}}(\mathbf{x}, t | (y, B_2), \tau) f_{(X_1^a, T_2)}(y, \tau) dy d\tau.
\end{aligned}$$

REMARK 3.2. *Assume to have a bivariate process with a component which is reset to its starting value whenever it crosses its boundary, and then it restarts. Meanwhile, the other component pursues its evolution. The intertimes between two consecutive FPTs of the same component are independent and identically distributed and thus each component is marginally modeled by a renewal process. The joint FPT distribution depends on $f_{(X_j^a, T_i)}$ and $f_{(X_i, T_j)}$. The second density describes the evolution of the process in (T_i, T_j) when $T_i < T_j$: \mathbf{X} starts in $\mathbf{X}(T_i) = (X_j^a(T_i), x_{0i})$, where the crossing component X_i is reset to its initial value x_{0i} . Then the slower component j has its FPT at time T_j , while X_i evolves, possibly with further crossings of B_i (with corresponding reset) before T_j , with $i, j = 1, 2$.*

4. Numerical method and its convergence property. For a diffusion process, the density $f_{(X_j^a, T_i)}$ is generally unknown. Therefore the system (3.8) cannot be analytically solved and neither the joint FPT distribution (3.1) nor (3.5) can be explicitly derived. To solve (3.8) and therefore approximate $f_{(X_j^a, T_i)}$, we propose a numerical method which does not depend on

whether the boundaries are crossing or absorbing. Consider an assigned two-dimensional time interval $[0, \Theta_1] \times [0, \Theta_2]$, with $\Theta_1, \Theta_2 \in \mathbb{R}^+$. For each component $i = 1, 2$, let h_i and r_i be the time and space discretization steps, respectively. On $\{[-\infty, B_1] \times [-\infty, B_2] \times [0, \Theta_1] \times [0, \Theta_2]\}$ we introduce the partition $\{(y_{u_1}, y_{u_2}); t_{k_1}, t_{k_2}\}$ where $t_{k_i} = k_i h_i$ is the time discretization and $y_{u_i} = B_i - u_i r_i$ is the space discretization for $k_i = 0, \dots, N_i, N_i h_i = \Theta_i, u_i \in \mathbb{N}$, and $i = 1, 2$. To simplify the notation, we consider $h_1 = h_2 = h$ and $\Theta_1 = \Theta_2 = \Theta$, implying $N_1 = N_2 = N$, $k_1 = k_2 = k$ and thus $t_{k_1} = t_{k_2} = t_k$, for $k = 0, \dots, N$.

Let $\hat{f}_{(X_1^a, T_2)}(y, t_j)$ denote the approximation of $f_{(X_1^a, T_2)}(y, t_j)$ due to the time discretization procedure. We approximate the time integrals of (3.8) through the Euler method [15], obtaining a system of integral equations for $\hat{f}_{(X_i^a, T_j)}(y, t)$, $i = 1, 2$. For $x_1 < B_1$ and $x_2 < B_2$, we get

$$(4.1a) \quad \begin{aligned} \bar{F}_{\mathbf{X}}((x_1, B_2), t_k) &= h \sum_{\rho=0}^k \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y), t_\rho) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\ &\quad + h \sum_{\rho=0}^k \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y, B_2), t_\rho) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy; \end{aligned}$$

$$(4.1b) \quad \begin{aligned} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k) &= h \sum_{\rho=0}^k \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y), t_\rho) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\ &\quad + h \sum_{\rho=0}^k \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y, B_2), t_\rho) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy. \end{aligned}$$

Let $\mathbb{1}_A$ denote the indicator function of the set A . Then

$$(4.2) \quad \begin{aligned} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y), t_k) &= \mathbb{1}_{\{y > x_2\}}; \\ \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y, B_2), t_k) &= 0; \\ \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y), t_k) &= 0; \\ \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y, B_2), t_k) &= \mathbb{1}_{\{y > x_1\}}. \end{aligned}$$

Plugging (4.2) into (4.1) and differentiating with respect to $x_j, j = 1, 2$, we get the system

$$\begin{aligned}
 (4.3a) \quad \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k)}{\partial x_1} &= h \hat{f}_{(X_1^a, T_2)}(x_1, t_k) \\
 &+ h \sum_{\rho=0}^{k-1} \int_{-\infty}^{B_2} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y), t_\rho)}{\partial x_1} \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\
 &+ h \sum_{\rho=0}^{k-1} \int_{-\infty}^{B_1} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y, B_2), t_\rho)}{\partial x_1} \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy;
 \end{aligned}$$

$$\begin{aligned}
 (4.3b) \quad \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k)}{\partial x_2} &= h \hat{f}_{(X_2^a, T_1)}(x_2, t_k) \\
 &+ h \sum_{\rho=0}^{k-1} \int_{-\infty}^{B_2} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y), t_\rho)}{\partial x_2} \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\
 &+ h \sum_{\rho=0}^{k-2} \int_{-\infty}^{B_1} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y, B_2), t_\rho)}{\partial x_2} \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy.
 \end{aligned}$$

Discretizing the spatial integral and truncating the corresponding series with a finite sum, we obtain

$$\begin{aligned}
 (4.4a) \quad \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k)}{\partial x_1} &= h \tilde{f}_{(X_1^a, T_2)}(x_1, t_k) \\
 &+ hr_2 \sum_{\rho=0}^{k-1} \sum_{u_2=0}^{m_2} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y_{u_2}), t_\rho)}{\partial x_1} \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_\rho) \\
 &+ hr_1 \sum_{\rho=0}^{k-1} \sum_{u_1=0}^{m_1} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y_{u_1}, B_2), t_\rho)}{\partial x_1} \tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_\rho);
 \end{aligned}$$

$$\begin{aligned}
 (4.4b) \quad \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k)}{\partial x_2} &= h \tilde{f}_{(X_2^a, T_1)}(x_2, t_k) \\
 &+ hr_2 \sum_{\rho=0}^{k-1} \sum_{u_2=0}^{m_2} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y_{u_2}), t_\rho)}{\partial x_2} \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_\rho) \\
 &+ hr_1 \sum_{\rho=0}^{k-1} \sum_{u_1=0}^{m_1} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y_{u_1}, B_2), t_\rho)}{\partial x_2} \tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_\rho).
 \end{aligned}$$

Here $\tilde{f}_{(X_i^a, T_j)}(y, t)$ denotes the approximation of $f_{(X_i^a, T_j)}(y, t)$ due to the time and space discretization procedures and to the truncation of the infinite sums of the space discretization.

Since $f_{(X_i^a, T_j)}(y_{u_i}, t_0) = 0$, we set $\tilde{f}_{(X_i, T_j)}(y_{u_i}, t_0) = 0$. The following algorithm can be used to approximate $f_{(X_i^a, T_j)}$ in the knots $\{(y_{u_1}, y_{u_2}); t_k\}$:

Step 1

$$\begin{aligned}\tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_1) &= \frac{1}{h} \frac{\partial}{\partial x_1} \bar{F}_{\mathbf{X}}((x_1, B_2), t_1) \Big|_{x_1=y_{u_1}}; \\ \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_1) &= \frac{1}{h} \frac{\partial}{\partial x_2} \bar{F}_{\mathbf{X}}((B_1, x_2), t_1) \Big|_{x_2=y_{u_2}}.\end{aligned}$$

Step $k \geq 2$

$$\begin{aligned}\tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_k) &= \left[\frac{1}{h} \frac{\partial}{\partial x_1} \bar{F}_{\mathbf{X}}((x_1, B_2), t_k) \Big|_{x_1=y_{u_1}} \right. \\ &- r_2 \sum_{\rho=0}^{k-1} \sum_{v_2=0}^{m_2} \tilde{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \frac{\partial}{\partial x_1} [\bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y_{v_2}), t_\rho)] \Big|_{x_1=y_{u_2}} \\ &- \left. r_1 \sum_{\rho=0}^{k-1} \sum_{v_1=0}^{m_1} \tilde{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \frac{\partial}{\partial x_1} [\bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y_{v_1}, B_2), t_\rho)] \Big|_{x_1=y_{u_1}} \right]; \\ \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_k) &= \left[\frac{1}{h} \frac{\partial}{\partial x_2} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k) \Big|_{x_2=y_{u_2}} \right. \\ &- r_2 \sum_{\rho=0}^{k-1} \sum_{v_2=0}^{m_2} \tilde{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \frac{\partial}{\partial x_2} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y_{v_2}), t_\rho) \Big|_{x_2=y_{u_2}} \\ &- \left. r_1 \sum_{\rho=0}^{k-1} \sum_{v_1=0}^{m_1} \tilde{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \frac{\partial}{\partial x_2} \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y_{v_1}, B_2), t_\rho) \Big|_{x_2=y_{u_1}} \right].\end{aligned}$$

Note that at time t_1 , $\hat{f}_{(X_i^a, T_j)}(y, t_1) = \tilde{f}_{(X_i^a, T_j)}(y_{u_i}, t_1)$ in each knot y_{u_i} .

REMARK 4.1. *We choose the Euler method because it simplifies the notation and is easy to implement. More efficient schemes, e.g. trapezoidal formula, can be similarly applied, improving the rate of convergence error of the proposed algorithm.*

5. Convergence of the algorithm. Let $E^{(i)}(y_{u_i}, t_k)$ denote the error of the proposed algorithm evaluated in the mesh points (y_{u_i}, t_k) , for $k = 0, \dots, N, u_i = 0, 1, \dots, m_i, i = 1, 2$. It is defined by

$$(5.1) \quad E^{(i)}(y_{u_i}, t_k) = f_{(X_i^a, T_j)}(y_{u_i}, t_k) - \tilde{f}_{(X_i^a, T_j)}(y_{u_i}, t_k), \quad i, j = 1, 2, i \neq j.$$

Mimicking the analysis of the error in [8], we rewrite the error (5.1) as a sum of two errors. The first is given by $e_k^{(i)}(y_{u_i}) = f_{(X_i^a, T_j)}(y_{u_i}, t_k) - \hat{f}_{(X_i^a, T_j)}(y_{u_i}, t_k)$ and is due to the time discretization.

The second is given by $E_{k,u_i}^{(i)} = \hat{f}_{(X_i^a, T_j)}(y_{u_i}, t_k) - \tilde{f}_{(X_i^a, T_j)}(y_{u_i}, t_k)$ and is determined by the spatial discretization and by the truncation introduced at steps $k \geq 2$. We start computing $E_{k,u_i}^{(i)}$ through the following

LEMMA 5.1. *It holds*

$$\begin{aligned}
 E_{k,u_1}^{(1)} = & \sum_{\rho=0}^{k-1} \left[- \int_{-\infty}^{B_1} K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2)) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\
 & - \int_{-\infty}^{B_2} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y)) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\
 & + r_1 \sum_{v_1=0}^{m_1} K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2)) \tilde{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \\
 & \left. + r_2 \sum_{v_2=0}^{m_2} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y_{v_2})) \tilde{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \right];
 \end{aligned}
 \tag{5.2a}$$

$$\begin{aligned}
 E_{k,u_2}^{(2)} = & \sum_{\rho=0}^{k-1} \left[- \int_{-\infty}^{B_1} K_{2,k,\rho}((B_1, y_{u_2}), (y, B_2)) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\
 & - \int_{-\infty}^{B_2} K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y)) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\
 & + r_1 \sum_{v_1=0}^{m_1} K_{2,k,\rho}((B_1, y_{u_2}), (y_{v_1}, B_2)) \tilde{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \\
 & \left. + r_2 \sum_{v_2=0}^{m_2} K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y_{v_2})) \tilde{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \right],
 \end{aligned}
 \tag{5.2b}$$

where the kernels are

$$\begin{aligned}
 K_{1,k,t}((y_{u_1}, b), (c, d)) &= \frac{\partial}{\partial x_1} \left[\bar{F}_{\mathbf{X}}((x_1, b), t_k | (c, d), t) - \bar{F}_{\mathbf{X}}((x_1, b), t_{k-1} | (c, d), t) \right] \Big|_{x_1=y_{u_1}} \\
 K_{2,k,t}((a, y_{u_2}), (c, d)) &= \frac{\partial}{\partial x_2} \left[\bar{F}_{\mathbf{X}}((a, x_2), t_k | (c, d), t) - \bar{F}_{\mathbf{X}}((a, x_2), t_{k-1} | (c, d), t) \right] \Big|_{x_2=y_{u_2}}.
 \end{aligned}
 \tag{5.3}$$

When $t = t_\rho$, we write $K_{i,k,\rho}$ instead of K_{i,k,t_ρ} to simplify the notation. Here $a, c \in (-\infty, B_1)$ and $b, d \in (-\infty, B_2)$.

PROOF. Subtracting (4.4) from (4.3), we obtain

$$\begin{aligned}
 \hat{f}_{(X_1^a, T_2)}(x_1, t_k) - \tilde{f}_{(X_1^a, T_2)}(x_1, t_k) &= \sum_{\rho=0}^{k-1} \left[- \int_{-\infty}^{B_1} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y, B_2), t_\rho)}{\partial x_1} \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\
 &\quad - \int_{-\infty}^{B_2} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y), t_\rho)}{\partial x_1} \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\
 &\quad + r_1 \sum_{u_1=0}^{m_1} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y_{u_1}, B_2), t_\rho)}{\partial x_1} \tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_\rho) \\
 &\quad \left. + r_2 \sum_{u_2=0}^{m_2} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y_{u_2}), t_\rho)}{\partial x_1} \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_\rho) \right]; \tag{5.4a}
 \end{aligned}$$

$$\begin{aligned}
 \hat{f}_{(X_2^a, T_1)}(x_2, t_k) - \tilde{f}_{(X_2^a, T_1)}(x_2, t_k) &= \sum_{\rho=0}^{k-1} \left[- \int_{-\infty}^{B_1} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y, B_2), t_\rho)}{\partial x_2} \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\
 &\quad - \int_{-\infty}^{B_2} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y), t_\rho)}{\partial x_2} \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\
 &\quad + r_1 \sum_{\rho=0}^k \sum_{u_1=0}^{m_1} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y_{u_1}, B_2), t_\rho)}{\partial x_2} \tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_\rho) \\
 &\quad \left. + r_2 \sum_{u_2=0}^{m_2} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y_{u_2}), t_\rho)}{\partial x_2} \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_\rho) \right]. \tag{5.4b}
 \end{aligned}$$

If we rewrite (5.4) for $k - 1$, without making the $(k - 1)$ th term explicit, we obtain

$$(5.5a) \quad \begin{aligned} & \sum_{\rho=0}^{k-1} \left[- \int_{-\infty}^{B_1} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y, B_2), t_\rho)}{\partial x_1} \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\ & - \int_{-\infty}^{B_2} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y), t_\rho)}{\partial x_1} \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\ & + r_1 \sum_{u_1=0}^{m_1} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (y_{u_1}, B_2), t_\rho)}{\partial x_1} \tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_\rho) \\ & \left. + r_2 \sum_{u_2=0}^{m_2} \frac{\partial \bar{F}_{\mathbf{X}}((x_1, B_2), t_k | (B_1, y_{u_2}), t_\rho)}{\partial x_1} \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_\rho) \right] = 0; \end{aligned}$$

$$(5.5b) \quad \begin{aligned} & \sum_{\rho=0}^{k-1} \left[- \int_{-\infty}^{B_1} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y, B_2), t_\rho)}{\partial x_2} \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\ & - \int_{-\infty}^{B_2} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y), t_\rho)}{\partial x_2} \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \\ & + r_1 \sum_{\rho=0}^k \sum_{u_1=0}^{m_1} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (y_{u_1}, B_2), t_\rho)}{\partial x_2} \tilde{f}_{(X_1^a, T_2)}(y_{u_1}, t_\rho) \\ & \left. + r_2 \sum_{u_2=0}^{m_2} \frac{\partial \bar{F}_{\mathbf{X}}((B_1, x_2), t_k | (B_1, y_{u_2}), t_\rho)}{\partial x_2} \tilde{f}_{(X_2^a, T_1)}(y_{u_2}, t_\rho) \right] = 0, \end{aligned}$$

due to conditions (4.2). Then, the result follows by subtracting (5.5) from (5.4) and setting $x_i = y_{u_i}$, for $i = 1, 2$. \square

To prove the convergence of the proposed algorithm, we need the following conditions:

CONDITION 5.1. For $i, j = 1, 2, i \neq j, k = 1, \dots, N, \rho = 0, \dots, k - 1, u_i = 0, \dots, m_i$:

- (i) $K_{i,k,\rho}((a, b), (c, y)) \hat{f}_{(X_j^a, T_i)}(y, t_\rho)$ and $K_{i,k,\rho}((a, b), (y, d)) \hat{f}_{(X_i^a, T_j)}(y, t_\rho)$ are ultimately monotonic in y ;
- (ii) $f_{(X_i^a, T_j)}(y, t_\rho)$ is bounded, belongs to L^1 and there exist positive functions $C_{i,1}(y) \in L^1$, $C_{i,2}(y) \in L^1$, with $x \in (-\infty, B_1)$ and $y \in (-\infty, B_2)$, such that

$$\begin{aligned} |K_{i,k,\rho}((a, b), (y, d))| & \leq h C_{i,1}(y); \\ |K_{i,k,\rho}((a, b), (c, y))| & \leq h C_{i,2}(y), \end{aligned}$$

and $C_{i,1}(0)$ and $C_{i,2}(0)$ are bounded;

(iii) for $l = 1, 2$

$$(5.6) \quad \int_{-\infty}^{B_i - m_i(r_i)r_i} C_{l,i}(y) \left| \hat{f}_{(X_i^a, T_j)}(y, t_\rho) \right| dy \leq \psi_{l,i} r_i,$$

as $r_i \rightarrow 0$ and $m_i(r_i)r_i \rightarrow \infty$, where $\psi_{l,i}$ are positive constants;

(iv) for $l = 1, 2$, there exist constants $Q_{l,i}$ such that

$$(5.7) \quad \left| \int_{-\infty}^{B_i} \frac{\partial}{\partial t} \left[C_{l,i}(y) f_{(X_i^a, T_j)}(y, t_\rho) \right] dy \right| \leq Q_{l,i};$$

(v) for $l = 1, 2$, $\mathbf{z}_1 = (y_{u_1}, B_2)$ and $\mathbf{z}_2 = (B_1, y_{u_2})$,

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\bar{F}_{\mathbf{X}}(\mathbf{z}_l, t_\rho | (B_1, y), t) f_{(X_2^a, T^1)}(y, t) \right] |_{t=\tau} \in L^1 \text{ in } y \in (-\infty, B_2); \\ & \frac{\partial}{\partial t} \left[\bar{F}_{\mathbf{X}}(\mathbf{z}_l, t_\rho | (y, B_2), t) f_{(X_1^a, T^2)}(y, t) \right] |_{t=\tau} \in L^1 \text{ in } y \in (-\infty, B_1); \\ & \frac{\partial}{\partial y_{u_l}} \frac{\partial}{\partial t} \left[\bar{F}_{\mathbf{X}}(\mathbf{z}_1, t_\rho | (B_1, y), t) f_{(X_2^a, T^1)}(y, t) \right] |_{t=\tau} \in L^1 \text{ in } y \in (-\infty, B_2); \\ & \frac{\partial}{\partial y_{u_l}} \frac{\partial}{\partial t} \left[\bar{F}_{\mathbf{X}}(\mathbf{z}_l, t_\rho | (y, B_2), t) f_{(X_1^a, T^2)}(y, t) \right] |_{t=\tau} \in L^1 \text{ in } y \in (-\infty, B_1). \end{aligned}$$

The following theorem gives the convergence of the proposed algorithm.

THEOREM 5.1. *If conditions 5.1 are satisfied, then*

$$(5.8) \quad E^{(i)}(y_{u_i}, t_k) = O(h) + O(r),$$

where $r = \max(r_1, r_2)$.

PROOF. At first, we study the error $E_{k,u_i}^{(i)}$ due to the spatial discretization. It can be decomposed as

$$(5.9) \quad E_{k,u_i}^{(i)} = A_{k,u_i}^{(i)} - B_{k,u_i}^{(i)}, \quad k = 1, \dots, N.$$

Here, $A_{k,u_i}^{(i)}$ has the same expression of $E_{k,u_i}^{(i)}$ in (5.2), replacing $\tilde{f}_{(X_i^a, T_j)}(y, t_j)$ with $\hat{f}_{(X_i^a, T_j)}(y, t_j)$.

Moreover, $B_{k,u_i}^{(i)}$ is defined by

$$(5.10a) \quad B_{k,u_1}^{(1)} = \sum_{\rho=0}^{k-1} \left[r_1 \sum_{v_1=0}^{m_1} K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2)) E_{\rho,v_1}^{(1)} \right. \\ \left. + r_2 \sum_{v_2=0}^{m_2} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y_{v_2})) E_{\rho,v_2}^{(2)} \right],$$

$$(5.10b) \quad B_{k,u_2}^{(2)} = \sum_{\rho=0}^{k-1} \left[r_1 \sum_{v_1=0}^{m_1} K_{2,k,\rho}((B_1, y_{u_2}), (y_{v_1}, B_2)) E_{\rho,v_1}^{(1)} \right. \\ \left. + r_2 \sum_{v_2=0}^{m_2} K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y_{v_2})) E_{\rho,v_2}^{(2)} \right].$$

The term $A_{k,u_i}^{(i)}$ accounts for the approximation of the spatial integrals with finite sums. Hence we can split it into a first term $A_{k,u_i}^{(i,a)}$ accounting for the discretization procedure and a second $A_{k,u_i}^{(i,b)}$ for the truncation of the series.

By definition of $A_{k,u_i}^{(i,a)}$, we have

$$\begin{aligned}
 |A_{k,u_1}^{(1,a)}| &= \left| \sum_{\rho=0}^{k-1} \left\{ \left[\int_{-\infty}^{B_1} K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2)) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \right. \right. \\
 &\quad \left. \left. - r_1 \sum_{v_1=0}^{\infty} K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2)) \hat{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \right] \right. \\
 &\quad \left. + \left[\int_{-\infty}^{B_2} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y)) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \right. \right. \\
 &\quad \left. \left. - r_2 \sum_{v_2=0}^{\infty} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y_{v_2})) \hat{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \right] \right\} \Bigg|; \tag{5.11a}
 \end{aligned}$$

$$\begin{aligned}
 |A_{k,u_2}^{(2,a)}| &= \left| \sum_{\rho=0}^{k-1} \left\{ \left[\int_{-\infty}^{B_1} K_{2,k,\rho}((B_1, y_{u_2}), (y, B_2)) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \right. \right. \\
 &\quad \left. \left. - r_1 \sum_{v_1=0}^{\infty} K_{2,k,\rho}((B_1, y_{u_2}), (y_{v_1}, B_2)) \hat{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \right] \right. \\
 &\quad \left. + \left[\int_{-\infty}^{B_2} K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y)) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \right. \right. \\
 &\quad \left. \left. - r_2 \sum_{v_2=0}^{\infty} K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y_{v_2})) \hat{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \right] \right\} \Bigg|. \tag{5.11b}
 \end{aligned}$$

Let us focus on the terms in the first square brackets in (5.11a). It holds

$$\begin{aligned}
 &\left| \int_{-\infty}^{B_1} K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2)) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \\
 &\quad \left. - r_1 \sum_{v_1=0}^{\infty} K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2)) \hat{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \right| \\
 &\leq \left| \int_{B_1-r_1}^{B_1} K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2)) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right| \\
 &\leq h \int_{B_1-r_1}^{B_1} C_{1,1}(y) \left| \hat{f}_{(X_1^a, T_2)}(y, t_\rho) \right| dy \\
 &\leq hr_1 \eta_{1,1}, \tag{5.12}
 \end{aligned}$$

where we used condition 5.1(i) and eq. (3.4.5) in [10] in the first inequality and condition 5.1(ii) in the second. Note that the numerical approximations $\hat{f}_{(X_j^a, T_i)}(y, t_\rho)$ can be rewritten as a function

of $f_{(X_j^a, T_i)}(y, t_\rho)$ and $K_{i,k,\rho}(a, b, c, d)$, as shown in Remark 2 in [8]. Then, thanks to condition 5.1(ii), it follows that $\hat{f}_{(X_j^a, T_i)}(y, t_\rho)$ is bounded. Moreover, the integrable function $C_{1,1}(y)$ on the compact interval $[B_1 - r_1, B_1]$ is bounded. Thus $C_{1,1}(y)|\hat{f}_{(X_1^a, T_2)}(y, t_\rho)| \leq \eta_{1,1}$ for a positive constant η_1 , which yields (5.12). A similar procedure can be done for the terms in the second square brackets in (5.11a) and for those in (5.11b), obtaining

$$(5.13a) \quad |A_{k,u_1}^{(1,a)}| \leq (r_1\eta_{1,1} + r_2\eta_{1,2}) \sum_{\rho=0}^{k-1} h = (r_1\eta_{1,1} + r_2\eta_{1,2})t_{k-1};$$

$$(5.13b) \quad |A_{k,u_2}^{(2,a)}| \leq (r_1\eta_{2,1} + r_2\eta_{2,2}) \sum_{\rho=0}^{k-1} h = (r_1\eta_{2,1} + r_2\eta_{2,2})t_{k-1}.$$

Here $\eta_{i,l}$ are positive constants given by

$$C_{l,i}(y)f_{(X_i^a, T_j)}(y, t_\rho) \leq \eta_{l,i},$$

for $i, j, l = 1, 2, i \neq j$. Let us now consider the error $A_{k,u_i}^{(i,b)}$. Using condition 5.1(i), eq. (3.4.5) in [10] and then conditions 5.1(ii), 5.1(iii) in sequence, we get

$$\begin{aligned} |A_{k,u_1}^{(1,b)}| &= \left| \sum_{\rho=0}^{k-1} \left[r_1 \sum_{v_1=m_1+1}^{\infty} K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2)) \hat{f}_{(X_1^a, T_2)}(y_{v_1}, t_\rho) \right. \right. \\ &\quad \left. \left. + r_2 \sum_{v_2=m_2+1}^{\infty} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y_{v_2})) \hat{f}_{(X_2^a, T_1)}(y_{v_2}, t_\rho) \right] \right| \\ &\leq \left| \sum_{\rho=0}^{k-1} h \left[\int_{-\infty}^{B_1-m_1r_1} C_{1,1}(y) \hat{f}_{(X_1^a, T_2)}(y, t_\rho) dy \right. \right. \\ &\quad \left. \left. + \int_{-\infty}^{B_2-m_2r_2} C_{1,2}(y) \hat{f}_{(X_2^a, T_1)}(y, t_\rho) dy \right] \right| \\ (5.14a) \quad &\leq (\psi_{1,1}r_1 + \psi_{1,2}r_2)t_k; \end{aligned}$$

$$(5.14b) \quad |A_{k,u_2}^{(2,b)}| \leq (\psi_{2,1}r_1 + \psi_{2,2}r_2)t_k,$$

where (5.14b) is obtained as (5.14a).

From (5.13), (5.14) and $r = \max(r_1, r_2)$, we get $|A_{k,u_i}^{(i)}| \leq rG_it_k$, where G_i , $i = 1, 2$ are positive suitable constants. Using these bounds in (5.9) and observing that $B_{k,u_i}^{(i)}$ in (5.10) involves the

errors $E_{\rho,v_i}^{(i)}$ for $0 \leq \rho \leq k-1$, we get a system of inequalities

$$(5.15a) \quad |E_{k,u_1}^{(1)}| \leq G_1 r t_k + r \sum_{\rho=0}^{k-1} \left[\sum_{v_1=0}^{m_1} |K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2))| |E_{\rho,v_1}^{(1)}| \right. \\ \left. + \sum_{v_2=0}^{m_2} |K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y_{v_2}))| |E_{\rho,v_2}^{(2)}| \right]$$

$$(5.15b) \quad |E_{k,u_2}^{(2)}| \leq G_2 r t_k + r \sum_{\rho=0}^{k-1} \left[\sum_{v_1=0}^{m_1} |K_{2,k,\rho}((B_1, y_{u_2}), (y_{v_1}, B_2))| |E_{\rho,v_1}^{(1)}| \right. \\ \left. + \sum_{v_2=0}^{m_2} |K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y_{v_2}))| |E_{\rho,v_2}^{(2)}| \right].$$

We extend the method proposed in [8] to the system (5.15), that we solve iteratively as follows:

$$(5.16) \quad \begin{aligned} |E_{0,u_i}^{(i)}| &= 0 := r p_0^{(i)}; \\ |E_{1,u_i}^{(i)}| &\leq G_i r t_1 =: r p_1^{(i)}; \\ |E_{2,u_1}^{(1)}| &\leq G_1 r t_2 + r \left[r p_1^{(1)} \sum_{v_1=0}^{m_1} |K_{1,k,\rho}((y_{u_1}, B_2), (y_{v_1}, B_2))| \right. \\ &\quad \left. + r p_1^{(2)} \sum_{v_2=0}^{m_2} |K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y_{v_2}))| \right] \\ &\leq r \left[G_1 t_2 + r \beta_1^1 p_1^{(1)} + r \beta_2^1 p_1^{(2)} \right] =: r p_2^{(1)}; \\ |E_{2,u_2}^{(2)}| &\leq r \left[G_2 t_2 + r \beta_1^2 p_1^{(1)} + r \beta_2^2 p_1^{(2)} \right] =: r p_2^{(2)}, \end{aligned}$$

where (5.16) holds due to condition 5.1(ii) and eq. (3.4.5) in [10]. Here $\beta_l^i, i, l = 1, 2$ are suitable constants, which do not depend on r and h . Iterating this procedure, (5.15) becomes

$$(5.17a) \quad |E_{k,u_1}^{(1)}| \leq r \left[G_1 t_k + r \sum_{\rho=0}^{k-1} \left(\beta_1^1 p_\rho^{(1)} + \beta_2^1 p_\rho^{(2)} \right) \right] =: r p_k^{(1)};$$

$$(5.17b) \quad |E_{k,u_2}^{(2)}| \leq r \left[G_2 t_k + r \sum_{\rho=0}^{k-1} \left(\beta_1^2 p_\rho^{(1)} + \beta_2^2 p_\rho^{(2)} \right) \right] =: r p_k^{(2)}.$$

Since $t_k \leq \Theta$, from (5.17) it follows

$$p_k^{(i)} \leq G_i \Theta + r \sum_{\rho=0}^{k-1} \left(\beta_1^i p_\rho^{(1)} + \beta_2^i p_\rho^{(2)} \right), \quad i = 1, 2.$$

Then, by eq. (7.18) in [15], we get $p_k^{(i)} \leq G_i \Theta \exp[(\beta_1^1 + \beta_2^2)rt_k]$. Therefore

$$(5.18a) \quad |E_{k,u_1}^{(1)}| \leq rG_1 \Theta \exp[(\beta_1^1 + \beta_2^1)rt_k];$$

$$(5.18b) \quad |E_{k,u_2}^{(2)}| \leq rG_2 \Theta \exp[(\beta_1^2 + \beta_2^2)rt_k],$$

implying $|E_{k,u_i}^{(i)}| = O(r)$.

Now, we focus on the time discretization error $e_k^{(i)}(y_{u_i})$. The error formulas for the Euler method are

$$(5.19) \quad \begin{aligned} \delta_{1,1,k}(h) &= \frac{ht_k}{2} \int_{-\infty}^{B_1} \frac{\partial}{\partial t} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k | (y, B_2), t) f_{(X_1^a, T_2)}(y, t) dy \Big|_{t=\tau}; \\ \delta_{1,2,k}(h) &= \frac{ht_k}{2} \int_{-\infty}^{B_2} \frac{\partial}{\partial t} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k | (B_1, y), t) f_{(X_2^a, T_1)}(y, t) dy \Big|_{t=\tau}; \\ \delta_{2,1,k}(h) &= \frac{ht_k}{2} \int_{-\infty}^{B_1} \frac{\partial}{\partial t} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k | (y, B_2), t) f_{(X_1^a, T_2)}(y, t) dy \Big|_{t=\tau}; \\ \delta_{2,2,k}(h) &= \frac{ht_k}{2} \int_{-\infty}^{B_2} \frac{\partial}{\partial t} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k | (B_1, y), t) f_{(X_2^a, T_1)}(y, t) dy \Big|_{t=\tau}, \end{aligned}$$

where $\tau \in (0, \Theta)$ and we used $t_k = hk$. Rewriting (4.1) with the corresponding residuals and evaluating it in $x_i = y_{u_i}$, $i = 1, 2$, we get

$$(5.20a) \quad \begin{aligned} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k) &= h \sum_{\rho=0}^k \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k | (y, B_2), t_\rho) f_{(X_1^a, T_2)}(y, t_\rho) dy \\ &\quad + h \sum_{\rho=0}^k \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k | (B_1, y), t_\rho) f_{(X_2^a, T_1)}(y, t_\rho) dy \\ &\quad + \delta_{1,1,k}(h) + \delta_{1,2,k}(h); \end{aligned}$$

$$(5.20b) \quad \begin{aligned} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k) &= +h \sum_{\rho=0}^k \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k | (y, B_2), t_\rho) f_{(X_1^a, T_2)}(y, t_\rho) dy \\ &\quad + h \sum_{\rho=0}^k \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k | (B_1, y), t_\rho) f_{(X_2^a, T_1)}(y, t_\rho) dy \\ &\quad + \delta_{2,1,k}(h) + \delta_{2,2,k}(h). \end{aligned}$$

Subtracting (4.1) from (5.20) and differentiating with respect to y_{u_i} , we get a system of integral

equations for $e_\rho^{(i)}(y)$ given by

$$(5.21a) \quad -\frac{\partial}{\partial y_{u_1}} [\delta_{1,1,k}(h) + \delta_{1,2,k}(h)] = \left\{ h \sum_{\rho=0}^k \left[\frac{\partial}{\partial y_{u_1}} \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k | (y, B_2), t_\rho) e_\rho^{(1)}(y) dy \right. \right.$$

$$\left. + \frac{\partial}{\partial y_{u_1}} \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((y_{u_1}, B_2), t_k | (B_1, y), t_\rho) e_\rho^{(2)}(y) dy \right] \Big\};$$

$$(5.21b) \quad -\frac{\partial}{\partial y_{u_2}} [\delta_{2,1,k}(h) + \delta_{2,2,k}(h)] = \left\{ h \sum_{\rho=0}^k \left[\frac{\partial}{\partial y_{u_2}} \int_{-\infty}^{B_1} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k | (y, B_2), t_\rho) e_\rho^{(1)}(y) dy \right. \right.$$

$$\left. + \frac{\partial}{\partial y_{u_2}} \int_{-\infty}^{B_2} \bar{F}_{\mathbf{X}}((B_1, y_{u_2}), t_k | (B_1, y), t_\rho) e_\rho^{(2)}(y) dy \right] \Big\}.$$

Rewriting (5.21) with respect to $k-1$, subtracting it from (5.21) and using (4.2), we obtain

$$(5.22a) \quad \begin{aligned} e_k^{(1)}(y_{u_1}) - \sum_{\rho=0}^{k-1} & \left[\int_{-\infty}^{B_1} K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2)) e_\rho^{(1)}(y) dy \right. \\ & \left. + \int_{-\infty}^{B_2} K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y)) e_\rho^{(2)}(y) dy \right] \\ & = \frac{\partial}{\partial y_{u_1}} \left[\frac{(\delta_{1,1,k}(h) - \delta_{1,1,k-1}(h)) + (\delta_{1,2,k}(h) - \delta_{1,2,k-1}(h))}{h} \right]; \end{aligned}$$

$$(5.22b) \quad \begin{aligned} e_k^{(2)}(y_{u_2}) - \sum_{\rho=0}^{k-1} & \left[\int_{-\infty}^{B_1} K_{2,k,\rho}((B_1, y_{u_2}), (y, B_2)) e_\rho^{(1)}(y) dy \right. \\ & \left. + \int_{-\infty}^{B_2} K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y)) e_\rho^{(2)}(y) dy \right] \\ & = \frac{\partial}{\partial y_{u_2}} \left[\frac{(\delta_{2,1,k}(h) - \delta_{2,1,k-1}(h)) + (\delta_{2,2,k}(h) - \delta_{2,2,k-1}(h))}{h} \right]. \end{aligned}$$

Using (5.3), (5.19) and condition 5.1(v), and since $t_{k-1} = t_k - h$, we have

$$\begin{aligned}
& \left| \frac{\partial}{\partial y_{u_1}} |\delta_{1,1,k}(h) - \delta_{1,1,k-1}(h)| \right| \\
& \leq \frac{ht_k}{2} \int_{-\infty}^{B_1} \left| \frac{\partial}{\partial t} \left[|K_{1,k,t}((y_{u_1}, B_2), (y, B_2))| |f_{(X_1^a, T_2)}(y, t)| dy \right] \right|_{t=\tau} \\
& + \frac{h^2}{2} \left| \frac{\partial}{\partial y_{u_1}} \int_{-\infty}^{B_1} \frac{\partial}{\partial t} \left[\bar{F}_X((y_{u_1}, B_2), t_{k-1}|(y, B_2), t) f_{(X_1^a, T_2)}(y, t) dy \right] \right|_{t=\tau} \\
& \leq \frac{h^2}{2} [t_k Q_{1,1} + S_{1,1}] := \frac{h^2}{2} \alpha_{1,1}.
\end{aligned}$$

The last inequality holds applying conditions 5.1(ii) and 5.1(iv) on the first term, and condition 5.1(v) on the second term, for a suitable positive constant $S_{1,1}$. Similarly

$$\left| \frac{\partial}{\partial y_{u_l}} |\delta_{l,i,k}(h) - \delta_{l,i,k-1}(h)| \right| \leq \frac{h^2}{2} [t_k Q_{l,i} + S_{l,i}] := \frac{h^2}{2} \alpha_{l,i},$$

for $i, l = 1, 2$ and suitable positive constants $S_{l,i}$ obtained from condition 5.1(v). Then (5.22) becomes

$$\begin{aligned}
(5.23a) \quad |e_k^{(1)}(y_{u_1})| & \leq \frac{(\alpha_{1,1} + \alpha_{1,2})ht_k}{2} + \sum_{\rho=0}^{k-1} \left[\int_{-\infty}^{B_1} |K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2)) e_\rho^{(1)}(y)| dy \right. \\
& \quad \left. + \int_{-\infty}^{B_2} |K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y)) e_\rho^{(2)}(y)| dy \right];
\end{aligned}$$

$$\begin{aligned}
(5.23b) \quad |e_k^{(2)}(y_{u_2})| & \leq \frac{(\alpha_{2,1} + \alpha_{2,2})ht_k}{2} + \sum_{\rho=0}^{k-1} \left[\int_{-\infty}^{B_1} |K_{2,k,\rho}((B_1, y_{u_2}), (y, B_2)) e_\rho^{(1)}(y)| dy \right. \\
& \quad \left. + \int_{-\infty}^{B_2} |K_{2,k,\rho}((B_1, y_{u_2}), (B_1, y)) e_\rho^{(2)}(y)| dy \right].
\end{aligned}$$

Setting $\gamma_l = \max\{\alpha_{l,1}, \alpha_{l,2}\}$, for $l = 1, 2$, we can write the system (5.23) iteratively for $k \geq 0$, obtaining

$$\begin{aligned}
|e_0^{(i)}| & = 0 := hq_0^{(i)}; \\
|e_1^{(i)}| & \leq \gamma_i ht_1 := hq_1^{(i)}; \\
|e_2^{(1)}| & \leq \gamma_1 ht_2 + hq_1^{(1)} \int_{-\infty}^{B_1} |K_{1,k,\rho}((y_{u_1}, B_2), (y, B_2))| dy + hq_1^{(2)} \int_{-\infty}^{B_2} |K_{1,k,\rho}((y_{u_1}, B_2), (B_1, y))| dy \\
& \leq h \left(\gamma_1 t_2 + h\xi_1^1 q_1^{(1)} + h\xi_2^1 q_1^{(2)} \right) := hq_2^{(1)}; \\
|e_2^{(2)}| & \leq h \left(\gamma_2 t_2 + h\xi_1^2 q_1^{(1)} + h\xi_2^2 q_1^{(2)} \right) := hq_2^{(2)},
\end{aligned}$$

where we used condition 5.1(ii) to bound $e_2^{(i)}$. Here ξ_i^j are suitable constants independent on h and r . In general

$$(5.24a) \quad |e_k^{(1)}(y_{u_1})| \leq h \left[\gamma_1 t_k + h \left(\xi_1^1 \sum_{\rho=0}^{k-1} q_\rho^{(1)} + \xi_2^1 \sum_{\rho=0}^{k-1} q_\rho^{(2)} \right) \right] := h q_k^{(1)};$$

$$(5.24b) \quad |e_k^{(2)}(y_{u_2})| \leq h \left[\gamma_2 t_k + h \left(\xi_1^2 \sum_{\rho=0}^{k-1} q_\rho^{(1)} + \xi_2^2 \sum_{\rho=0}^{k-1} q_\rho^{(2)} \right) \right] := h q_k^{(2)}.$$

Since $t_k \leq \Theta$, from (5.24) it follows

$$q_k^{(i)} \leq \gamma_i \Theta + h \left(\xi_1^i \sum_{\rho=0}^{k-1} q_\rho^{(1)} + \xi_2^i \sum_{\rho=0}^{k-1} q_\rho^{(2)} \right), \quad i = 1, 2,$$

and applying eq. (7.18) in [15], we get $q_k^{(i)} \leq \gamma_i \Theta \exp[(\xi_1^i + \xi_2^i)t_k]$ and thus

$$(5.25a) \quad |e_k^{(1)}(y_{u_1})| \leq h \gamma_1 \Theta \exp[(\xi_1^1 + \xi_2^1)t_k];$$

$$(5.25b) \quad |e_k^{(2)}(y_{u_2})| \leq h \gamma_2 \Theta \exp[(\xi_1^2 + \xi_2^2)t_k].$$

The result follows noting that $|e_k^{(i)}(y_{u_i})| = O(h)$. \square

REMARK 5.1. *The numerical approximations $\hat{f}_{(X_j^a, T_i)}(y, t_\rho)$ can be rewritten such that they depend only on $f_{(X_j^a, T_i)}(y, t_\rho)$ and $K_{i,k,\rho}(a, b, c, d)$, as shown in Remark 2 in [8]. Therefore, conditions 5.1(i) and 5.1(iii) are in fact assumptions on f and K .*

6. Joint distribution of (T_1, T_2) for a bivariate Wiener process. Consider a bivariate Wiener process \mathbf{X} solving (2.1) with constant drift $\boldsymbol{\mu}(\mathbf{X}(t)) = (\mu_1, \mu_2) \in \mathbb{R}^2$ and positive-definite covariance matrix

$$\boldsymbol{\Sigma}(\mathbf{X}(t), t) = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix},$$

for $\rho \in (-1, 1)$. Then \mathbf{X} is a bivariate Wiener process with null drift and covariance matrix

$$\tilde{\boldsymbol{\Sigma}} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

That is, each component is marginally a Wiener process with drift μ_i , diffusion coefficient $\sigma_i > 0$, $i = 1, 2$ and ρ is the correlation of the bivariate Wiener process, e.g. $\rho = 0$ corresponds to have independent components. For the Wiener process, the densities $f_{\mathbf{X}}$, $f_{X_i^a}$ and g_{T_i} , $i, j = 1, 2$, $i \neq j$ are known [9]. To apply Theorems 3.1 and 3.2, we need to calculate the unknown density $f_{(X_j^a, T_i)}$.

Since g_{T_i} is known, this corresponds to derive the conditional density $f_{X_j^a|T_i}$. The first step is to calculate the unknown transition density $f_{\mathbf{X}^a}$, which solves the two-dimensional Kolmogorov forward equation

$$(6.1) \quad \begin{aligned} \frac{\partial f_{\mathbf{X}^a}(\mathbf{x}, t)}{\partial t} &= \frac{\sigma_1^2}{2} \frac{\partial^2 f_{\mathbf{X}^a}(\mathbf{x}, t)}{\partial x_1^2} + \frac{\sigma_2^2}{2} \frac{\partial^2 f_{\mathbf{X}^a}(\mathbf{x}, t)}{\partial x_2^2} + \sigma_1 \sigma_2 \rho \frac{\partial^2 f_{\mathbf{X}^a}(\mathbf{x}, t)}{\partial x_1 \partial x_2} \\ &- \mu_1 \frac{\partial f_{\mathbf{X}^a}(\mathbf{x}, t)}{\partial x_1} - \mu_2 \frac{\partial f_{\mathbf{X}^a}(\mathbf{x}, t)}{\partial x_2}, \end{aligned}$$

with initial, boundary and absorbing conditions given by

$$(6.2) \quad \lim_{t \rightarrow 0} f_{\mathbf{X}^a}(\mathbf{x}, t) = \delta(x_1 - x_{01}) \delta(x_2 - x_{02});$$

$$(6.3) \quad \lim_{x_1 \rightarrow -\infty} f_{\mathbf{X}^a}(\mathbf{x}, t) = \lim_{x_2 \rightarrow -\infty} f_{\mathbf{X}^a}(\mathbf{x}, t) = 0;$$

$$(6.4) \quad f_{\mathbf{X}^a}(\mathbf{x}, t)|_{x_1=B_1} = f_{\mathbf{X}^a}(\mathbf{x}, t)|_{x_2=B_2} = 0,$$

respectively, where we set $t_0 = 0$. The solution provided in [12] does not fulfill (6.2) when $(\mu_1, \mu_2) \neq (0, 0)$. Following their proof, we noted that the normalizing factor

$$(6.5) \quad \exp \left(- \frac{(\mu_2 \rho \sigma_1 \sigma_2 - \mu_1 \sigma_2^2) B_1 + (\mu_1 \rho \sigma_1 \sigma_2 - \mu_2 \sigma_1^2) B_2}{(1 - \rho^2) \sigma_1^2 \sigma_2^2} \right)$$

was missing. Since (6.5) is equal to 1 when $(\mu_1, \mu_2) = (0, 0)$, the results in [12] are correct for the driftless case. In presence of drift, it holds

LEMMA 6.1. *The density $f_{\mathbf{X}^a}$ that the process never reaches the boundary \mathbf{B} in $(0, t)$ is given by*

$$(6.6) \quad \begin{aligned} f_{\mathbf{X}^a}(\mathbf{x}, t) &= \frac{2}{\alpha K_3 t} \exp \{K_1(B_1 - x_{01}) + K_2(B_2 - x_{02})\} \\ &\times \exp \left(- \frac{\sigma_1^2 \mu_2^2 - 2\mu_1 \mu_2 \sigma_1 \sigma_2 \rho + \sigma_2^2 \mu_1^2}{2K_3^2} t - \frac{\bar{r}^2 + \bar{r}_0^2}{2K_3^2 t} \right) H(\bar{r}, \bar{r}_0, \phi, \phi_0, t), \end{aligned}$$

where $\bar{r} := \bar{r}(x_1, x_2) \in (0, \infty)$, $\phi := \phi(x_1, x_2) \in (0, \alpha)$ and

$$\begin{aligned}
\bar{r} &= \sqrt{\sigma_1^2(B_2 - x_2)^2 + \sigma_2^2(B_1 - x_1)^2 - 2\sigma_1\sigma_2\rho(B_1 - x_1)(B_2 - x_2)}; \\
\bar{r} \cos(\phi) &= \sigma_2(B_1 - x_1) - \sigma_1\rho(B_2 - x_2), \quad \bar{r} \sin(\phi) = \sigma_1\sqrt{1 - \rho^2}(B_2 - x_2); \\
\bar{r}_0 &= \bar{r}|_{x_1=x_{01}; x_2=x_{02}}; \\
\phi_0 &= \phi|_{x_1=x_{01}; x_2=x_{02}}; \\
K_1 &= \frac{\sigma_2\mu_1 - \sigma_1\mu_2\rho}{\sigma_1^2\sigma_2(1 - \rho^2)}, \quad K_2 = \frac{\sigma_1\mu_2 - \sigma_2\mu_1\rho}{\sigma_1\sigma_2^2(1 - \rho^2)}, \quad K_3 = \sigma_1\sigma_2\sqrt{1 - \rho^2}; \\
\alpha &= \arctan\left(-\frac{\sqrt{1 - \rho^2}}{\rho}\right) \in (0, \pi); \\
H(\bar{r}, \bar{r}_0, \phi, \phi_0, t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\phi_0}{\alpha}\right) \sin\left(\frac{n\pi\phi}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{\bar{r}\bar{r}_0}{K_3^2 t}\right).
\end{aligned}$$

Here (\bar{r}, ϕ) are functions of (x_1, x_2) and are obtained through a suitable change of variables in [12]. In (6.6) we use them instead of (x_1, x_2) to simplify the notation. To compare our results with those in [12], one should introduce the transformation $r = \bar{r}/\sigma_1$ and $r_0 = \bar{r}_0/\sigma_1$, since they use different constant terms.

REMARK 6.1. *The distribution of the first exit time of the process from the strip $(-\infty, B_1) \times (-\infty, B_2)$ is given by*

$$\mathbb{P}(\min(T_1, T_2) < t) = 1 - \int_{-\infty}^{B_1} \int_{-\infty}^{B_2} f_{\mathbf{X}^a}(\mathbf{x}, t) dx_1 dx_2,$$

and it can be computed multiplying eq. (32) in [12] with the missing factor (6.5).

COROLLARY 6.1. *The conditional density $f_{X_i^a|X_j^a}(x_i, t|x_j, t)$, for $i, j = 1, 2$; $i \neq j$ is given by*

$$\begin{aligned}
f_{X_i^a|X_j^a}(x_i, t|x_j, t) &= \frac{2\sigma_j\sqrt{2\pi t}}{\alpha K_3 t} \exp\left(-K_i \left[\frac{\sigma_i}{\sigma_j}(x_j - x_{0j})\rho - (x_i - x_{0i})\right]\right) \\
&\times \exp\left(-K_i t N_j - \frac{\bar{r}^2 + \bar{r}_0^2 - (x_j - x_{0j})^2 \sigma_i^2 (1 - \rho^2)}{2K_3^2 t}\right) \\
(6.7) \quad &\times \left[1 - \exp\left(\frac{2(B_j - x_{0j})(x_j - B_j)}{\sigma_j^2 t}\right)\right]^{-1} H(\bar{r}, \bar{r}_0, \phi, \phi_0, t),
\end{aligned}$$

with

$$N_1 = \frac{\sigma_1\mu_2 - \sigma_2\mu_1\rho}{2\sigma_1}, \quad N_2 = \frac{\sigma_2\mu_1 - \sigma_1\mu_2\rho}{2\sigma_2}.$$

PROOF. The conditional density is given by

$$(6.8) \quad f_{X_i^a|X_j^a}(x_i, t, |x_j, t; \mathbf{y}, s) = \frac{f_{\mathbf{X}^a}(\mathbf{x}, t | \mathbf{y}, s)}{f_{X_j^a}(x_j, t | y_j, s)},$$

and the result follows by plugging $f_{X_j^a}$ given in [20] and (6.6) into (6.8). \square

Then, we introduce the following

LEMMA 6.2. *The conditional density $f_{X_i^a|T_j}(x_i|t)$ for $i, j = 1, 2$ $i \neq j$ is given by*

$$(6.9) \quad f_{X_i^a|T_j}(x_i|t) = \frac{\sigma_j \pi \sqrt{2\pi t}}{\alpha^2 (B_i - x_i)(B_j - x_{0j})} \exp \left(-K_i \left[\frac{\sigma_i}{\sigma_j} (B_j - x_{0j}) \rho - (x_i - x_{0i}) \right] \right) \\ \times \exp \left(-K_i t N_j - \frac{[\rho \sigma_i (B_j - x_{0j}) - \sigma_j (B_i - x_{0i})]^2 + \sigma_j^2 (B_i - x_i)^2}{2K_3^2 t} \right) G_{ij}(\bar{r}_0, \phi_0, x_i, t),$$

where

$$G_{ij}(\bar{r}_0, \phi_0, x_i, t) = \sum_{n=1}^{\infty} \delta_i n \sin \left(\frac{n\pi \phi_0}{\alpha} \right) I_{\frac{n\pi}{\alpha}} \left(\frac{\sigma_j (B_i - x_i) \bar{r}_0}{K_3^2 t} \right),$$

with $\delta_1 = 1$ and $\delta_2 = (-1)^{n+1}$.

PROOF. When $x_j \rightarrow B_j$, both $f_{X_j^a}$ and $f_{\mathbf{X}^a}$ go to zero, due to the boundary condition (6.4). Therefore $f_{X_i^a|X_j^a}$ is indefinite, as noticed from (6.8). From the definition of ϕ , we have that $\phi \rightarrow \alpha$ when $x_1 \rightarrow B_1$ and $\phi \rightarrow 0$ when $x_2 \rightarrow B_2$. In both cases, $\sin(n\pi\phi/\alpha) \rightarrow 0$ and thus $H(\bar{r}, \bar{r}_0, \phi, \phi_0, t) \rightarrow 0$. Moreover,

$$\left[1 - \exp \left(\frac{2(B_j - x_{0j})(x_j - B_j)}{\sigma_j^2 t} \right) \right] \rightarrow 0,$$

when $x_j \rightarrow B_j$. Hence, the last two terms in (6.7) produce an indefinite form. Applying l'Hôpital's rule, we obtain

$$\lim_{x_j \rightarrow B_j} \frac{\sin \left(\frac{n\pi \phi}{\alpha} \right)}{1 - \exp \left(\frac{2(B_j - x_{0j})(x_j - B_j)}{\sigma_j^2 t} \right)} = \frac{\sigma_i \sigma_j \pi \sqrt{1 - \rho^2} t}{2\alpha (B_i - x_i)(B_j - x_{0j})} n \delta_i,$$

with $\delta_1 = 1, \delta_2 = (-1)^{n+1}$, depending on whether $\phi \rightarrow \alpha$ or $\phi \rightarrow 0$. The result follows by plugging this ratio into (6.7). \square

In presence of absorbing boundaries, the joint distribution of the FPTs can be explicitly calculated as follows

THEOREM 6.1. *The joint distribution of (T_1, T_2) in presence of absorbing boundaries is given by*

$$\begin{aligned}
 \mathbb{P}(T_1 < t_1, T_2 < t_2) &= \sum_{i,j=1;i \neq j}^2 \frac{\sqrt{2\pi}}{2\alpha^2\sigma_j} \exp\left(K_i(B_i - x_{0i}) - K_j x_{0j} + \frac{\mu_j B_j}{\sigma_j^2}\right) \\
 (6.10) \quad &\times \int_0^{t_i} \int_{-\infty}^{B_j} \int_{s_i}^{t_j} \frac{1}{s_i \sqrt{(s_j - s_i)^3}} \exp\left(-K_i \rho \frac{\sigma_i}{\sigma_j} x_j - K_i N_j s_i\right) \\
 &\times \exp\left(-\frac{\bar{r}_0^2 + \sigma_i^2(B_j - x_j)^2}{2s_i K_3^2} - \frac{(B_j - x_j)^2}{2\sigma_j^2(s_j - s_i)} - \frac{\mu_j^2 s_j}{2\sigma_j^2}\right) G_{ji}(\bar{r}_0, \phi_0, x_j, s_i) ds_j dx_j ds_i.
 \end{aligned}$$

PROOF. It follows by plugging g_{T_i} given in [20] and (6.9) into (3.1), and then simplifying the resulting expression. \square

6.1. *Driftless Wiener process.* For a driftless Wiener process in presence of absorbing boundaries, it holds

THEOREM 6.2. *The joint density of (T_1, T_2) in presence of absorbing boundaries is given by*

$$\begin{aligned}
 \mathbb{P}(T_1 \in dt_1, T_2 \in dt_2) &= \sum_{i,j=1;i \neq j}^2 \frac{\pi \sqrt{1 - \rho^2}}{2\alpha^2 \sqrt{t_i(t_j - t_i \rho^2)}(t_j - t_i)} \exp\left(-\frac{\bar{r}_0^2[t_j + t_i(1 - 2\rho^2)]}{4K_3^2 t_i(t_j - t_i \rho^2)}\right) \\
 (6.11) \quad &\times \sum_{n=1}^{\infty} \delta_j n \sin\left(\frac{n\pi\phi_0}{\alpha}\right) I_{\frac{n\pi}{2\alpha}}\left(\frac{\bar{r}_0^2(t_j - t_i)}{4K_3^2 t_i(t_j - t_i \rho^2)}\right) dt_i dt_j.
 \end{aligned}$$

PROOF. Since $\mu_1 = \mu_2 = 0$, it follows that $K_1 = K_2 = 0$. Deriving (6.10) with respect to t_1

and t_2 , we have

$$\begin{aligned}
& \mathbb{P}(T_1 \in dt_1, T_2 \in dt_2) \\
&= \sum_{i,j=1;i \neq j}^2 \int_{-\infty}^{B_j} \left[\frac{\sqrt{2\pi}}{2\alpha^2 \sigma_j t_i \sqrt{(t_j - t_i)^3}} \exp \left(-\frac{\bar{r}_0^2 + \sigma_i^2 (B_j - x_j)^2}{2t_i K_3^2} - \frac{(B_j - x_j)^2}{2\sigma_j^2 (t_j - t_i)} \right) \right. \\
&\quad \left. \times G_{ji}(\bar{r}_0, \phi_0, x_j, t_i) \right] dx_j dt_i dt_j \\
&= \sum_{i,j=1;i \neq j}^2 \frac{\sqrt{2\pi} \exp \left\{ -\frac{\bar{r}_0^2}{2t_i K_3^2} \right\}}{2\alpha^2 \sigma_j t_i \sqrt{(t_j - t_i)^3}} \sum_{n=1}^{\infty} \delta_j n \sin \left(\frac{n\pi \phi_0}{\alpha} \right) \\
&\quad \times \int_{-\infty}^{B_j} \exp \left(-\frac{\sigma_i^2 (B_j - x_j)^2 (t_j - t_i \rho^2)}{2K_3^2 t_i (t_j - t_i)} \right) I_{\frac{n\pi}{\alpha}} \left(\frac{\sigma_i (B_j - x_j) \bar{r}_0}{K_3^2 t_i} \right) dx_j dt_i dt_j \\
&= \sum_{i,j=1;i \neq j}^2 \frac{\sqrt{2\pi} \exp \left(-\frac{\bar{r}_0^2}{2t_i K_3^2} \right)}{2\alpha^2 \sigma_i \sigma_j t_i \sqrt{(t_j - t_i)^3}} \sum_{n=1}^{\infty} \delta_j n \sin \left(\frac{n\pi \phi_0}{\alpha} \right) \\
(6.12) \quad & \times \int_0^{\infty} \exp \left(-\frac{h^2 (t_j - t_i \rho^2)}{2K_3^2 t_i (t_j - t_i)} \right) I_{\frac{n\pi}{\alpha}} \left(\frac{h \bar{r}_0}{K_3^2 t_i} \right) dh dt_i dt_j,
\end{aligned}$$

where the last equality is obtained through a change of coordinate, namely $h = \sigma_i (B_j - x_j)$. The integral in (6.12) can be solved using the identity [16]

$$(6.13) \quad \int_0^{\infty} e^{-\beta^2 h^2} I_{\nu}(\gamma h) dh = \frac{\sqrt{\pi}}{2\beta} \exp \left(\frac{\gamma^2}{8\beta^2} \right) I_{\nu/2} \left(\frac{\gamma^2}{8\beta^2} \right),$$

setting $\beta^2 = (t_j - t_i \rho^2) / (2K_3^2 t_i (t_j - t_i))$, $\gamma = \bar{r}_0 / (K_3^2 t_i)$ and $\nu = n\pi / \alpha$. The result follows after some computations. \square

When $t_1 = t_2$, we have the following

COROLLARY 6.2. *The joint FPT density in presence of absorbing boundaries when $t_1 = t_2 = t$ is*

$$\mathbb{P}(T_1 \in dt, T_2 \in dt) = \begin{cases} 0dt^2 & \text{if } \rho \in (-1, 0) \\ \infty dt^2 & \text{if } \rho \in (0, 1) \\ \frac{(B_1 - x_{01})(B_2 - x_{02})}{2\pi \sigma_i \sigma_j t^3} e^{-\frac{\sigma_2^2 (B_1 - x_{01})^2 + \sigma_1^2 (B_2 - x_{02})^2}{2\sigma_1^2 \sigma_2^2 t}} dt^2 & \text{if } \rho = 0 \end{cases}$$

PROOF. If $t_i < t_j$, set $z = t_j - t_i$, $i, j = 1, 2, i \neq j$. When $z \rightarrow 0$, the limit of (6.11) is indefinite, being of the form $I_{\nu}(z)/z$, with $\nu = n\pi / (2\alpha)$. Using the fact that $I_{\nu}(z) \sim (z/2)^{\nu} / \Gamma(\nu + 1)$ [2],

we get

$$(6.14) \quad \lim_{z \rightarrow 0} \frac{I_\nu(z)}{z} = \frac{1}{2^\nu \Gamma(\nu + 1)} z^{\nu-1} = \begin{cases} 0 & \text{if } \nu > 1 \\ \infty & \text{if } \nu < 1 \\ \frac{1}{2} & \text{if } \nu = 0 \end{cases}$$

Since $\alpha \in (0, \pi)$, then $\nu > 1$ for $n > 2$, and thus all addends in the series in (6.11) vanish for $n > 2$. For $n = 2$, $\delta_1 = 1, \delta_2 = -1$ and thus the term in (6.11) is null, being the two densities symmetric. Finally, when $n = 1$, from (6.14), definitions of ν and ρ , it follows that $\nu < 1 \Leftrightarrow \alpha \in (0, \pi/2) \Leftrightarrow \rho \in (-1, 0)$; $\nu = 1 \Leftrightarrow \alpha = \pi/2 \Leftrightarrow \rho = 0$ and $\nu > 1 \Leftrightarrow \alpha \in (\pi/2, \pi) \Leftrightarrow \rho \in (0, 1)$, where \Leftrightarrow denotes *if and only if*. The result follows plugging the resulting expression for (6.14) into (6.11). \square

REMARK 6.2. To compare (6.11) with the corresponding expression in [13] for $s = t_1 < t_2 = t$, we set

$$\tilde{r}_0 = \frac{\bar{r}_0}{K_3},$$

since different transformations are used. Since

$$\sqrt{1 - \rho^2} = \sin \alpha, \quad \rho^2 = \cos^2 \alpha, \quad 2(t - s\rho^2) = (t - s) + (t - s \cos 2\alpha),$$

we obtain

$$\begin{aligned} & \mathbb{P}(T_1 \in ds, T_2 \in dt, s < t) \\ &= \frac{\pi \sin \alpha}{2\alpha^2 \sqrt{s(t - s \cos^2 \alpha)}(t - s)} \exp \left(-\frac{\tilde{r}_0^2(t - s \cos 2\alpha)}{2s[(t - s) + (t - s \cos 2\alpha)]} \right) \\ & \times \sum_{n=1}^{\infty} (-1)^{n+1} n \sin \left(\frac{n\pi\phi_0}{\alpha} \right) I_{\frac{n\pi}{2\alpha}} \left(\frac{\tilde{r}_0^2(t - s)}{2s[(t - s) + (t - s \cos 2\alpha)]} \right) ds dt. \end{aligned}$$

The result differs from that in [13], that uses an incorrect identity for (6.13), as already discussed in [12].

Since the joint density $f_{(X_j, T_i)}$ is not explicitly known, Theorem 3.2 cannot be used to determine an analytical expression of the joint distribution of (T_1, T_2) in presence of crossing boundaries. However, $f_{(X_j, T_i)}$ can be numerically evaluated for bivariate Gaussian diffusion as described in [5].

7. Examples. Denote $f_{\mathbf{T}}$ the theoretical joint density of (T_1, T_2) and $\hat{f}_{\mathbf{T}}$ its numerical approximation obtained applying the proposed algorithm. Here we report a brief illustration of the joint FPT density for two-dimensional Wiener and OU processes in presence of absorbing boundaries. For a bivariate Wiener, the theoretical joint density $f_{\mathbf{T}}$ is given by (6.11) for the

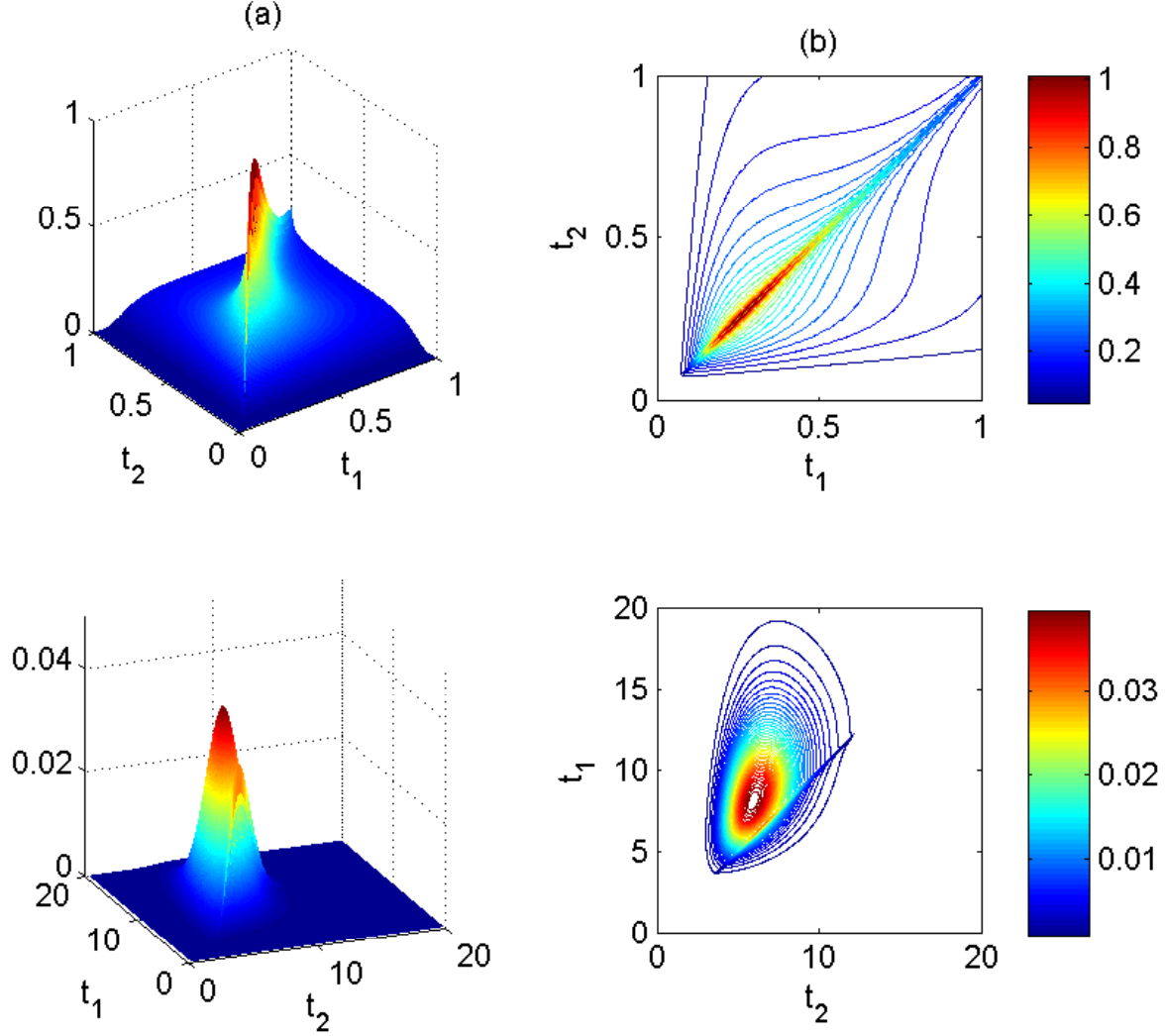


FIG 1. Theoretical joint densities and contour plots of (T_1, T_2) for two-dimensional Wiener processes in presence of absorbing boundaries. Parameter values common to all figures: $\sigma_1 = \sigma_2 = 1$, $\rho = 0.5$. Chosen drifts for the top figures: $\mu_1 = \mu_2 = 0$, $B_1 = B_2 = 1$ and time discretization step $h = 0.005$. Chosen drifts for the bottom figures: $\mu_1 = 1$, $\mu_2 = 1.5$, $B_1 = B_2 = 10$ and time discretization step $h = 0.05$. Panel (a): joint density of (T_1, T_2) . Panel (b): contour plots of (T_1, T_2) .

driftless case, and is derived from (6.10) when the drift is not null. To compare $f_{\mathbf{T}}$ and $\hat{f}_{\mathbf{T}}$, throughout we consider the mean square error (MSE), which is defined by

$$\text{MSE}(f_{\mathbf{T}}) = \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \left(f_{\mathbf{T}}(t_i, t_j) - \hat{f}_{\mathbf{T}}(t_i, t_j) \right)^2.$$

7.1. Bivariate Wiener process. First, consider a symmetric bivariate Wiener process with null drifts, parameters $\sigma_1 = \sigma_2 = 1$, $\rho = 0.5$ and boundaries $B_1 = B_2 = 1$. The theoretical joint density and its contour plot are reported in the top panels of Fig. 1. The numerical approximations are not shown, since they are indistinguishable from the theoretical. Indeed, choosing a space discretization step $r = 0.05$ and time discretization step $h = 0.01$ (resp. $h = 0.05$), we obtain $\text{MSE}(f_{\mathbf{T}}) = 3.5859 \cdot 10^{-5}$ (resp. $\text{MSE}(f_{\mathbf{T}}) = 4.8607 \cdot 10^{-4}$). This confirms the reliability of the algorithm, as expected from the convergence results of the error, proved in Theorem 5.1. Not surprisingly, also the joint FPT density and the contour plots are symmetric, and the probability mass is concentrated in the area close to the diagonal $t_1 = t_2$, representing simultaneous FPTs, i.e. $T_1 = T_2$.

Second, consider a non-symmetric bivariate Wiener process with parameters $\mu_1 = 1, \mu_2 = 2, \sigma_1 = \sigma_2 = 1, \rho = 0.5$ and boundaries $B_1 = B_2 = 10$. The joint density $f_{\mathbf{T}}$ and the contour plot are reported in the bottom panels of Fig. 1. They are indistinguishable from those obtained applying the numerical algorithm (figures not shown). As expected, the joint FPT density is not symmetric and the probability mass is concentrated around the means of the FPTs and it is spread out according to the variance of the FPTs. Indeed, for this parameter choice, we have

7.2. Bivariate Ornstein-Uhlenbeck process. A bivariate OU process satisfies the stochastic differential equation (2.1) with

$$(7.1) \quad \mu(\mathbf{X}(t)) = \begin{pmatrix} \mu_1 - \frac{X_1(t)}{\theta} \\ \mu_2 - \frac{X_2(t)}{\theta} \end{pmatrix}, \quad \Sigma(\mathbf{X}(t), t) = \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$

for $\mu_i \in \mathbb{R}, \sigma_{ij} > 0, 1 \leq i, j \leq 2, \sigma_{12} \in \mathbb{R}$ and Σ a positive-definite matrix.

Throughout we fix $\theta = 10, \sigma_{12} = 1, \sigma_i = 2, B_i = 10$, for $i = 1, 2$. First, we consider a symmetric OU with $\mu_1 = \mu_2 = 1.5$. The approximated joint density $\hat{f}_{\mathbf{T}}$ and the contour plot of (T_1, T_2) are given in the top panels of Fig. 2. With this parameter choice, the asymptotic mean $\mu_i \theta$ of each component of the OU is above the boundary B_i . Also in this case, the probability mass is concentrated along the diagonal $t_1 = t_2$. Hence, the times when the components cross their boundary are similar.

Second, we consider a non-symmetric OU with drifts $\mu_1 = 0.95$ and $\mu_2 = 1.5$. The approximated joint density $\hat{f}_{\mathbf{T}}$ and the contour plot are reported in the bottom panels of Fig. 2. Note that the first component has asymptotic mean $\mu_1 \theta$ below B_1 , and thus the noise determines the crossings of the boundary. As a consequence, the probabilistic mass is concentrated in the region $t_1 > t_2$.

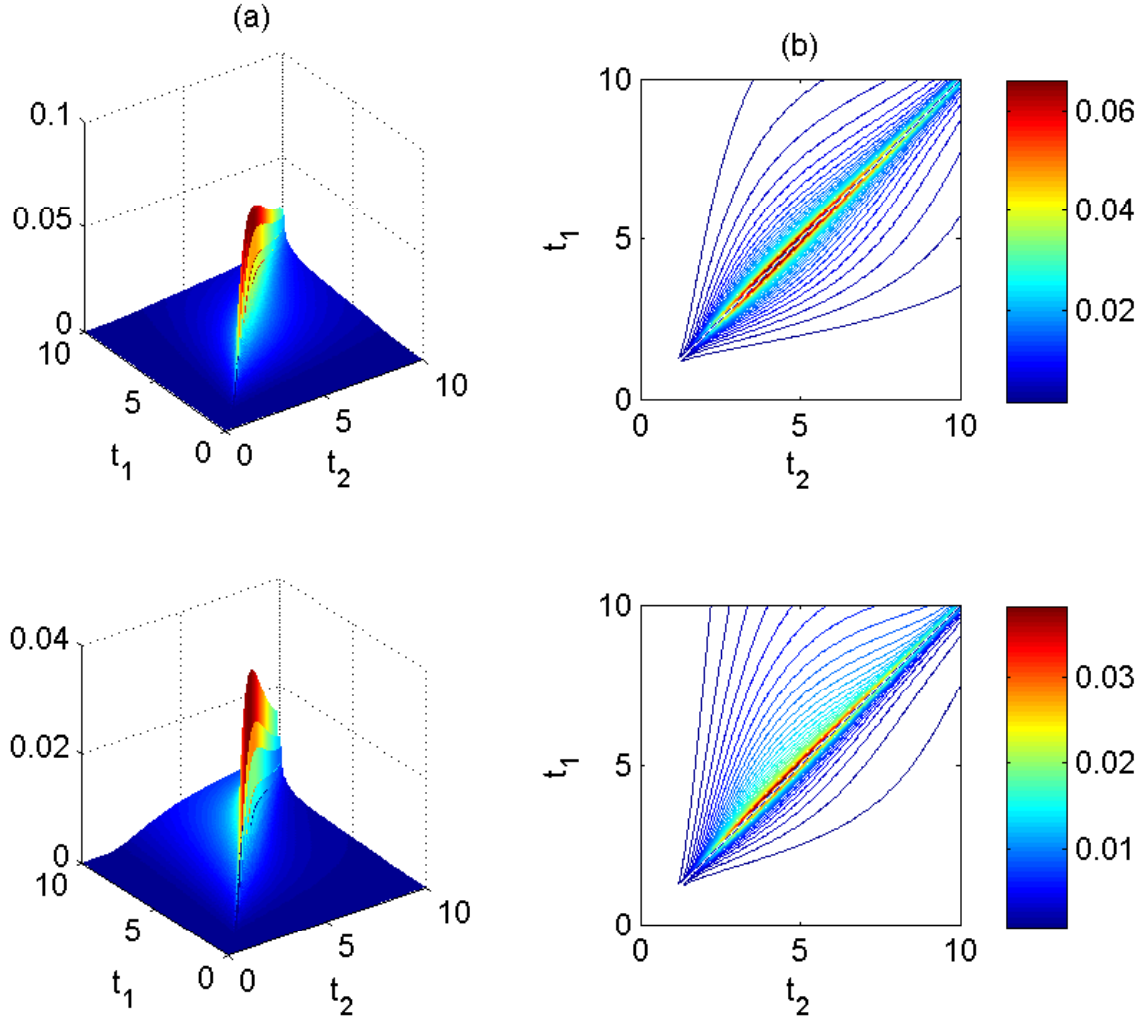


FIG 2. *Approximated joint densities and contour plots of (T_1, T_2) for bivariate OU processes in presence of absorbing boundaries. Parameter values common to all figures: $\sigma_{11} = \sigma_{22} = 2, \sigma_{12} = 1, B_1 = B_2 = 10$. Chosen drifts for the top figures: $\mu_1 = \mu_2 = 1.5$. Chosen drifts for the bottom figures: $\mu_1 = 0.95$ and $\mu_2 = 1.5$. Panel (a): joint density of (T_1, T_2) . Panel (b): contour plots of (T_1, T_2) .*

8. Conclusion. We solve the FPT problem for a two-dimensional Wiener process with constant drifts and non-diagonal covariance matrix in presence of absorbing boundaries. In particular, we explicitly calculate the joint density of the FPTs and other relevant quantities, e.g. the transition density of the process under the boundary. For bivariate diffusion processes in

presence of either absorbing or crossing boundaries, explicit expressions of those densities are not available. Therefore, we suggest to use the proposed numerical method for the evaluation of $f_{(X_1^a, T_2)}$ and $f_{(X_2^a, T_1)}$. The algorithm does not depend on whether the boundaries are crossing or absorbing, and its error is shown to converge. These results are also confirmed by our numerical examples for the Wiener case. Using the algorithms proposed here for $f_{(X_i^a, T_j)}$ and in [5] for $f_{(X_j, T_i)}$, it is possible to numerically evaluate the joint FPT density, provided that the bivariate transition density is known, e.g. for bivariate Gaussian diffusion processes. Furthermore, the proposed approach can be extended to the FPT problem for multivariate renewal processes with crossing boundaries, as argued in Remark 3.2.

A generalization to the joint FPT distribution of a k -dimensional process would also be of interest. However, that study requests the knowledge of the solution of a k -dimensional Kolmogorov forward equation, when the process is a multivariate Wiener in presence of absorbing boundaries, or of a system of k Volterra-Fredholm first kind integral equations, when the process is a diffusion.

Finally, note that both the theoretical results in Section 3 and the numerical algorithm can be extended to the case of time dependent boundaries.

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